THE AUCTION MODEL WITH LOWEST RISK IN A DUOPOLISTIC ELECTRICITY MARKET

ESTRELLA ALONSO Y JUAN TEJADA

RESUMEN

Este artículo modela la subasta del mercado eléctrico como un juego de dos jugadores de información incompleta bajo las hipótesis de empresas generadoras simétricas, neutrales al riesgo y con costes de producción independientes y privados. En Alonso and Tejada (2010) se define una amplia familia paramétrica de modelos de subasta que contiene a los modelos clásicos de subasta: uniforme, discriminatorio y de Vickrey. En el presente artículo se analiza la familia paramétrica de modelos de subasta mencionada desde el punto de vista del riesgo. Se diseña un modelo de subasta nuevo que se llamará DV, cuyo riesgo es más bajo que el obtenido con cualquiera de los modelos clásicos.
Clasificación JEL: D44
Palabras Clave: subasta, VaR, mercado eléctrico.

ABSTRACT

The present paper models the electricity market auction as a two-person game with incomplete information under the assumption that bidders are symmetric, risk neutral and have independent private costs. Alonso and Tejada (2010) define an extensive parametric family of auction models which contains the classic auction models; Uniform, Discriminatory and Vickrey auction models. The present paper analyzes this parametric family of auction models from the viewpoint of the risk. It develops a new auction model called DV, which has lower risk than any other classic auction model.
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Keywords: auctions, Value at Risk, electricity market.

THE AUCTION MODEL WITH LOWEST RISK IN A DUOPOLISTIC ELECTRICITY MARKET[1]

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I. Introduction

In many countries an auction model is used to regulate the daily Electricity Market (Fehr and Harbord, 1993). Each power company (female suppliers) bids an amount of electricity units and a unit price for each hour (or half hour) of the following day. From the supply side, the Market Operator (male auctioneer) ranks the bids from the lowest to the highest and then distributes the demand among the lowest bids, until the demand has been fully met. The price paid to each company taking part in the dispatch of the demand, depends on the auction model adopted for the transaction. There are two main auction models: the Uniform auction model and the Discriminatory auction model.

Much debate has been going on about the advantages and disadvantages of these auction models (Ausubel and Cramton, 2002, Fabra, 2001, Fabra, et al., 2002, Fabra et al., 2003), but no clear conclusion has been reached.

In the literature, there are papers that argue in favour of the Uniform model (Wolfram, 1999) whilst others favour the Discriminatory model (Federico and Rahman, 2001). There are other auction models used in contexts outside the Electricity Market, such as the Vickrey auction model (Vickrey, 1961).

One of the main difficulties is the fact that more than one object is being auctioned (the demand may be distributed among several companies) and the comparison between multiple-unit auction models is a very complex task. In single-unit auction models the solution is provided by the Revenue Equivalence Theorem (Myerson, 1981). This theorem is not generally applicable when the players wish to obtain more than one object, as it happens in electricity auctions.

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Alonso and Tejada (2010) define an extensive parametric family of auction models, called General Auction Model (*GAM*), which contains the classic models mentioned above: Uniform, Discriminatory and Vickrey. They characterize the strictly monotone Bayesian Nash equilibrium in all auctions models belonging to *GAM* and they prove that *GAM* verifies the corresponding Revenue Equivalence Theorem.

There is a wealth of literature that focuses in the optimal choice or design of auctions, usually from the point of view of the expected revenue (see, for example, Myerson, 1981; Riley and Samuelson, 1981). For this reason, if the Market Operator only takes into account the expected revenue, then every auction mechanism satisfying the Revenue Equivalence Theorem is indifferent for him. Nevertheless, an auction mechanism has to be selected. Consequently, Market Operator should take into account other criteria when making his decision. For instance, the criteria taken into account could be collusion (Robinson, 1985), another possible criteria could be variability (Vickrey, 1961), and other criteria could be used. Regarding the variability criterion, Vickrey (1961) calculates the variance for first-price and English auctions. Waehrer, et al. (1998) prove that a risk-averse auctioneer prefers first-price auction to second-price auction, and second-price auction to English auction. Beltrán and Santamaría (2006) use a simulation to analyze the variation for several auction mechanisms with the same expected revenue. Krishna (2002) proves that the price distribution in second-price auction is a mean-preserving spread of the price distribution in first-price auction.

The present paper starts from this point and develops a second criterion for the Market Operator that enables him to choose an appropriate auction model within the family. Therefore, the original contribution of the present paper is the analysis of risk measures applied to auction models potentially useful in the Electricity Market.

Variance is not the only indicator of the sign of payment deviations whilst payment can be volatile and reach sudden high values. As the Market Operator is not affected if payment to suppliers is lower than expected, the present paper intends in obtaining a measure for the risk of losses for each auction model too. Given that loss, in this context, is to pay more than expected to suppliers.

In this sense, the value at risk (VaR) measures the worst loss at a given confidence level and reflects how much can be lost with respect to expected at a certain probability (Holton, 2004). VaR is a risk measure used by financial institutions to study the market risk of their portfolios (Longerstaey and

Zangari, 1996). As previously stated, VaR could be very useful for helping the auctioneer to identify which auction mechanisms could be better when auctions are identical in terms of expected revenue but different in terms of risk losses. To establish a preference between the auction models, the variance and the VaR for Market Operator payment can be calculated in all auctions belonging to *GAM* and it can be found that under some conditions there are new auction models which have lower VaR than any other classic auction model.

On the other hand, in almost all the papers related to the Electricity Market, the theoretical model is considered as a non-cooperative game with complete information (where the production costs of the suppliers are considered public knowledge). There are few articles that assume the existence of some uncertainty on the costs function of a rival company, all of which use the Uniform auction model; Ferrero et al. (1998) present a numerical example in a duopolistic market. Their paper considers that the costs are private but correlated (production costs are private information of each company, but both depend on the price of fuel). Bosco and Parisio (2001) study an electricity market with N suppliers, each with two production units. Their paper considers that companies are asymmetrical in terms of production capacity, but assumes that the Bayesian Nash equilibria are symmetrical. Schöne (2003) considers a duopolistic market with identical capacities.

The present paper considers that bidders are symmetric and there exists some uncertainty on the cost function of the rival company, i.e., it models the electricity market auction as a game with incomplete information. This hypothesis assumes that the range of costs may be known for each plant type, but usually not all factors determining production costs are common knowledge.

The rest of the paper is organized as follows. In Section 2 the theoretical model to be analyzed is described and defined. Section 3, introduces the Bayesian Nash equilibrium for each auction model in the parametric family- as a particular case of the result obtained in Alonso and Tejada (2010). Section 4 is devoted to comparing the auction models in the parametric family with respect to their variance and their VaR for Market Operator payment, respectively. Finally, Section 5 concludes.

II. The theoretical model

The following hypotheses in the market is assumed:

There are two risk neutral suppliers $i \in \{1,2\}$ competing to provide the electricity required and they have the same perfectly divisible capacity $k = 1$.

Let $\{\Theta_1, \Theta_2\}$ be two independent and identically distributed random variables each uniform on $[0; 1]$. Supplier i and only supplier i, observes the realization θ_i of Θ_i and it gathers the uncertainty that company j has about the production cost of company i. If supplier i dispatches the amount $\phi \in [0, 1]$ then her production cost is $\phi\theta_i$.

The demand of a period is price-inelastic, known with certainty and it is given by $n + \alpha$, where $n \epsilon \{1,2, \dots\}$ and $\alpha \epsilon \{0,1\}$. That is, the full capacity of n suppliers and one supplier more to satisfy residual demand α are needed to satisfy full demand.

Each supplier $i \epsilon \{1,2\}$ simultaneously and independently submits a bid $b_i \epsilon [0,1]$specifying the minimum unit-price offer at which she is willing to supply the whole of her capacity.

A strategy for supplier $i \epsilon \{1,2\}$ is a function $b_i(.) : [0,1] \to [0,1]$.

It can be assumed, given the symmetry of the suppliers, that $b_i(\theta_i) = b(\theta_i)$ is the bid function used in the equilibrium by both companies, where $b(\theta_i)$ is a strictly monotone and differentiable function.

Once the Market Operator has received the bids, he allocates the electricity distribution in such a manner that the lowest bidder will dispatch first. If the suppliers capacity is not enough to satisfy the entire demand, then the highest bidder will satisfy the residual demand.

All aspects of this game and the auction model used, are assumed to be common knowledge (but the private information of each supplier cost).

The price paid to each supplier depends on the auction model adopted for the transaction. There are three classic auction models: **Uniform auction model, Discriminatory auction model and Vickrey auction model**.

In the Uniform auction model, the unit-price received by supplier i is equal to the highest accepted bid. All bidders dispatching into the market receive the same unit-price.

In the Discriminatory auction model, the unit-price received by supplier i is equal to her own bid b_i. All bidders dispatching into the market could receive a different unit-price.

In the Vickrey auction model, the rule used by the Market Operator to establish the price is more complicated than in the previous two models. The unit price received by supplier i dispatching in the market is equal to the bid of the supplier that i displaces, i.e., the unit price of the electricity unit needed to cover the demand if i removes her bid.

The present paper discusses not only the three classic models, but in fact it also considers a parametric family of auction models which contains the three classic models as particular cases. This family has already been defined in Alonso and Tejada (2010) but for completeness it is also included in the present paper.

Given the two bids, b_i denotes the bid of supplier i and b_j denotes the rival bid of supplier i. Formally, the parametric family of auction models is a set of auction models whose profit function (revenue minus cost) for supplier i is given by:

$$B_i(\theta_i, b_i, b_j) = \begin{cases} (\gamma_i b_i + \delta_1 b_j + \varphi) - \phi_1 \theta_i & if \quad b_i < b_j \\ (\gamma_i b_i + \varphi) - \phi_2 \theta_i & if \quad b_i > b_j \end{cases} \qquad \textbf{(1)}$$

where $\gamma_1, \delta_1,, \gamma_2, \gamma_2, \phi_1, \phi_2 \in [0, \infty)$ are electricity units verifying $\gamma_1 + \delta_1 + \varphi = \emptyset_1$, $\gamma_2 + \varphi = \phi_2$. That is, if i´s bid is the lowest, then i dispatchesϕ_1 units and possibly receives different prices for the total units dispatched: $\gamma_1$units by the unit price b_i, δ_1 units by the unit price b_j and φ units by the unit price 1, hence receives more than or equal to ϕ_1 times her own bid. On the other hand, if i´s bid is the highest, then i dispatches ϕ_2 units and possibly receives different prices for the total units dispatched: γ_2, units by the unit price b_i and φ units by the unit price 1, hence receives more than or equal to ϕ_2 times her own bid.

The values of ϕ_1 *and* ϕ_2 are completely determined by size of the demand: if $n = 0$ then $(\phi_1, \phi_2) = (\alpha;\ 0)$, if n = 1 then $(\phi_1, \phi_2) = (1; \alpha)$ and, finally, if $n \geq 2$ then $(\phi_1, \phi_2) = (1;\ 1)$.

This family of auction models verifies two desirable principles for an auction model to avoid the winner´s curse (Thaler, 1988) and to ensure effciency Burguet (2000), respectively. First, the bid made by a company is the minimum unit price at which she is willing to supply the whole of her capacity and the Market Operator cannot pay a company a unit price lower than her own bid. Second, the company that has made the lowest bid enters the market first and if it cannot satisfy all the demand, then the other company enters the market to dispatch the residual demand.

Let´s refer to this parametric family of auction models as **General Auction Model** (*GAM*).

Clearly, if the values of the parameters $\gamma_1, \delta_1, \varphi, \gamma_2$ are fixed, then the auction model used for the transaction is completely determined. Therefore we analyze the following different size demand cases are analyzed:

Case 1. Both suppliers have enough capacity to supply the whole demand, i.e., $n = 0$. In this case, the bidder with the lowest bid is the only one to dispatch. Then $\phi_1 = \alpha$, $\phi_2 = 0$, $\gamma_2 = \varphi = 0$ and $\delta_1 = \alpha - \gamma_1$ and substituting into the expression 1, the *GAM* is reduced to a uni-parametric family with supplier i´s proft function:

$$B_i^{\gamma_i}(\theta_i, b_i, b_j) = \begin{cases} [\gamma_i b_i + (\alpha - \gamma_1) b_j] - \alpha\theta_i & if \quad b_i < b_j \\ 0 & if \quad b_i > b_j \end{cases} \qquad \gamma_1 \epsilon [0, \alpha] \qquad \textbf{(2)}$$

In particular, if $\gamma_1 = \alpha$, the Uniform and Discriminatory auction models (*U\D*) is obtained. They are the same in this case. If now $\gamma_1 = 0$ the Vickrey auction model (*V I*) is obtained.

Case 1 is, in fact, a particular case of single-unit (α) auction. Therefore, it is known beforehand that the Revenue Equivalence Theorem is applicable.

Case 2. The capacity of both suppliers is needed to satisfy demand but there is excess overall capacity, i.e., $n = 1$. In this case, the bidder with the lowest bid produces 1 and the other bidder produces the residual demand α. Then in this case $\phi_1 = 1, \phi_2 = \alpha, \varphi = \alpha - \gamma_2$ and $\delta_1 = 1 - \alpha - \gamma_1 + \gamma_2$ and substituting into the expression 1, the *GAM* is reduced to a bi-parametric family with supplier i´s profit function:

$$B_i^{(\gamma_1, \gamma_2)}(\theta_i, b_i, b_j) = \begin{cases} [\gamma_1 b_i + (1 - \alpha - \gamma_1 + \gamma_2) b_j + \alpha - \gamma_2] - \theta_i & if \quad b_i < b_j \\ (\gamma_2 b_i + \alpha - \gamma_2) - \alpha\theta_i & if \quad b_i > b_j \end{cases} \qquad \textbf{(3)}$$

where $\gamma_1 \epsilon [0, 1 - \alpha + \gamma_2]$ and $\gamma_2 \epsilon [0, \alpha]$.

In particular, if $\gamma_1 = 0$ and $\gamma_2 = \alpha$, the Uniform auction model (*UP*) is obtained. If $\gamma_1 = 1$ and $\gamma_2 = \alpha$, the Discriminatory auction model (*DP*) is obtained. Finally, if If $\gamma_1 = 0$ and $\gamma_2 = 0$, we obtain the Vickrey auction model (*VI*). On the fourth vertex, i.e. $\gamma_1 = 1 - \alpha$ and $\gamma_2 = 0$, there is a new auction model that can be called DV auction (*DV*). Supplier i's profit function in DV model is:

$$B_i^{(1-\alpha,0)}(\theta_i, b_i, b_j) = \begin{cases} [(1-\alpha)b_i + \alpha] - \theta_i & if \quad b_i < b_j \\ \alpha - \alpha\theta_i & if \quad b_i > b_j \end{cases} \tag{4}$$

That is, if i's bid is the lowest then i dispatches her full own capacity and Market Operator pays her $(1-\alpha)$ units by the unit price b_i and α units by the unit price 1. On the other hand, if i's bid is the highest, then i dispatches only α units and receives the unit price 1 for each dispatched units.

DV is an auction model where the Market Operator pays to both suppliers α units by the unit price 1 and pays to bidder with the lowest bid the remaining dispatched units $(1-\alpha)$ multiply by her bid. *DV* is a mixture between Discriminatory and Vickrey auctions.

Case 3. Demand exceeds overall capacity, i.e. $n \geq 2$. In this case there is no competition. Both companies have guaranteed the dispatch of their entire capacity. The revenue for each supplier is 1 and the Market Operator pays 2. Obviously this is a trivial case.

III. Bayesian Nash equilibrium

Alonso and Tejada (2010) obtain the Bayesian Nash equilibrium of each auction model belonging to the family *GAM* where θ_i is an independent realization of a continuous random variable $\Theta_i$with c.d.f. *F* and they proved the following proposition:

Proposition 1. If an auction model $S \in GAM$ is used, then there exists a unique symmetric bayesian Nash equilibrium $b^*(1)$= 1, given by:

$$b^*(\theta_i) = \theta_i + (\gamma_1 - (\gamma_1 - \gamma_2)F(\theta_i))^{-\frac{\phi_1 - \phi_2}{\gamma_1 - \gamma_2}} \int_{\theta_i}^{1} (\gamma_1 - (\gamma_1 - \gamma_2)F(t))^{-\frac{\phi_1 - \phi_2}{\gamma_1 - \gamma_2}} dt$$

If $\gamma_1 \neq \gamma_2$,

$$b^*(\theta_i) = \theta_i + e^{-\frac{\phi_1 - \phi_2}{\gamma_1}(1-F(\theta_i))} \int_{\theta_i}^{1} e^{-\frac{\phi_1 - \phi_2}{\gamma_1}(1-F(t))} dt$$

If $\gamma_1 = \gamma_2 \neq 0, and,$

$b^*(\theta_i) = \theta_i$

If $\gamma_1 = \gamma_2 = 0$

Proposition 1 reduces in the particular case with uniformly distributed random variable Θ_i in [0; 1], for each demand size, to:

$$b^*(\theta_i) = \frac{\gamma_1 + \alpha\theta_i}{\alpha + \gamma_1} \quad \textbf{(5)}$$

substituting $\phi_1 = \alpha,\ \phi_2 = 0, \gamma_2 = 0$ for the Case 1; and

$$b^*(\theta_i) = \begin{cases} \theta_i & \text{If } \gamma_1 = \gamma_2 = 0 \\ \theta_i + \frac{\gamma_1}{1-\alpha}\left(e^{\frac{1-\alpha}{\gamma_1}(1-\theta_i)} - 1\right) & \text{If } \gamma_1 = \gamma_2 \neq 0 \\ \frac{\gamma_1 + (1-\alpha)\theta_i - \gamma_2 \gamma_1^{\frac{1-\alpha}{\gamma_1 - \gamma_2}+1}(\gamma_1 - (\gamma_1 - \gamma_2)\theta_i)^{-\frac{1-\alpha}{\gamma_1 - \gamma_2}}}{1-\alpha+\gamma_1-\gamma_2} & \text{If } \gamma_1 \neq \gamma_2 \end{cases} \quad \textbf{(6)}$$

substituting $\phi_1 = 1,\ \phi_2 = \alpha$ for the Case 2.

Remark 2. Note that, in equilibrium, supplier *i* offers a bid higher than or equal to θ_i. In addition the Market Operator cannot pay a company a unit price lower than her own bid. Then a supplier never gets less than the cost of the amount that she produces, therefore the bidders do not incur in an ex post loss problem. A supplier playing the equilibrium gets a positive surplus winning in each case and in each auction model belong to *GAM*.

Remark 3. Other possibility could be to bid different prices for the different electricity units but (Janssen et al., 2010) show that simultaneous pooled auctions with multiple bids do not have efficient equilibria under certain assumptions. This does not happen in any auction model belonging to *GAM*.

Remark 4. The companies have the same expected revenue $P_i^S(\theta_i)$and the same payment the Market Operator expects to make P_{MO}^S, for every $S \in GAM$ (Alonso and Tejada, 2010). Moreover, the expected revenue of the companies and the payment the Market Operator expects to make are:

$$P_i^S(\theta_i) = \begin{cases} \frac{\alpha}{2}\left(1-\theta_i^2\right) & in\ Case\ 1 \\ \frac{1}{2}\left(1+\alpha-(1-\alpha)\theta_i^2\right) & in\ Case\ 2 \\ 1 & in\ Case\ 3 \end{cases} \quad \textbf{(7)}$$

$$P_{MO}^S = 2\int_0^1 P_i^S(\theta_i)d\theta_i = \begin{cases} \frac{2}{3}\alpha & in\ Case\ 1 \\ \frac{2}{3}(2\alpha+1) & in\ Case\ 2 \\ 2 & in\ Case\ 3 \end{cases} \quad \textbf{(8)}$$

$\forall S \in GAM$

IV. Variance and value risk

If the Market Operator observes only the payment he expects to make then every auction model is equivalent for him. However, the models can be different if one takes into account others criteria such as variability or risk for the Market Operator. In this Section we calculate the variance and the value at risk of the Market Operator payment to suppliers at equilibrium in any auction model $S \in GAM$ are calculated. The Market Operator payment is given by the following random variable:

$$X^S = \begin{cases} \gamma_1 b^*\left(\Theta_{(2)}\right) + (\alpha-\gamma_1)b^*(\Theta_{(1)}) & in\ Case\ 1 \\ \gamma_1 b^*\left(\Theta_{(2)}\right) + (1-\alpha-\gamma_1+2\gamma_2)b^*\left(\Theta_{(1)}\right) + 2(\alpha-\gamma_2) & in\ Case\ 2 \end{cases} \quad \textbf{(9)}$$

Where $\Theta_{(1)}$ is the highest and $\Theta_{(2)}$ is the second-highest order statistics of $\{\Theta_1, \Theta_2\}$. The Market Operator is given the opportunity to distinguish among the auction models by using these criteria.

As it was discussed in the Introduction a usual way of addressing this issue is by means of the variance for this. The main problem of the variance is that it does not take into consideration if payments are below or above the expected payment. It is not the same for the Market Operator to pay above the expected payment, than to pay below the expected payment. As the Market Operator is not affected if payment to suppliers is lower than expected, the present paper also focuses in obtaining a measure for the risk of losses for each auction model too. A widely used measure of the risk of loss in different economic contexts is the value at risk. See, for instance, Holton (2004). In the electrical context, it can be defined as follows: given confidence level β, VaR at β is the smallest number k such that, the probability that the difference between the actual payment and the expected payment is greater than k is less than 1-β, that is

$$VaR_{\beta}^{S} = inf\{k \in [0, \mathrm{E}(\mathrm{X}^2)]: \mathrm{P}(\mathrm{X}^{\mathrm{S}} - \mathrm{E}(\mathrm{X}^{\mathrm{S}}) < k) \geq \beta\} \tag{10}$$

Note that if $S \epsilon$ *GAM* then $E(X^S) = P_{MO}^S$ given in Remark 4. On the other hand, to avoid the effect of $E(X^S)$ we can use the relative VaR ($RVaR$) that is given by:

$$RVaR_{\beta}^{S} = \frac{VaR_{\beta}^{S}}{E(X^S)} x100 \tag{11}$$

The Market Operator pays a extra amount less than $RVaR_{\beta}^{S}$ % of the expected payment, with probability greater than β. With VaR and $RVaR$ we can measure how much the Market Operator can pay over expected payment at a certain probability.

IV.1. Variance in Case 1

The variance of the payment that the Market operator makes in Case 1 is given by:

$$\begin{aligned} V(X^S) &= E((X^S)^2) - \left(E(X^S)\right)^2 = \\ &= 2\int_0^1 \int_{\theta_i}^1 \left[\gamma_1 b^*(\theta_i) + (\alpha - \gamma_1) b^*(\theta_j)\right]^2 d\theta_j \theta_i - \left(\frac{2\alpha}{3}\right)^2 \\ &= \frac{\alpha^2(\alpha^2 - \alpha\gamma_1 + \gamma_1^2)}{18(\alpha + \gamma_1)^2} \end{aligned} \tag{12}$$

and replacing the corresponding value of γ_1 into the expression 12. Table 1 lists the variance associated to *VI* auction model and *U\D* auction model.

Table 1.
Variance of VI and U\D in Case 1

	$b^*(\theta_i)$	X^S	$V(X^S)$
VI	θ_i	$\alpha\ \Theta_{(1)}$	$\frac{\alpha^2}{18}$
$U\backslash D$	$\frac{1+\theta_i}{2}$	$\frac{\alpha\left(1+\Theta_{(2)}\right)}{2}$	$\frac{\alpha^2}{72}$

The model with the lowest variance is *U\D* auction model.

IV.2. Variance in Case 2

The variance of the payment that the Market Operator makes in Case 2 is given by:

$$V(X^S) = E((X^S)^2) - \left(E(X^S)\right)^2$$
$$= 2\int_0^1 \int_{\theta_i}^1 [\gamma_1 b^*(\theta_i) + (1-\alpha-\gamma_1+2\gamma_2)b^*(\theta_j) + 2(\alpha-\gamma_2)]^2 d\theta_j d\theta_i - \left(\frac{2}{3}(2\alpha+1)\right)^2 \quad \textbf{(13)}$$

and replacing the corresponding Bayesian Nash equilibrium into the expression 13, Table 2 presents the variance associated to four vertexes of the parametric family.

Table 2.
Variance of V I, DV , UP and DP in Case 2.

	$b^*(\theta_i)$	X^S	$V(X^S)$
VI	θ_i	$(1-\alpha)\Theta_{(1)} + 2\alpha$	$\frac{(1-\alpha)^2}{18}$
DV	$\frac{1+\theta_i}{2}$	$\frac{(1-\alpha)\Theta_{(2)}}{2} + \frac{1+3\alpha}{2}$	$\frac{(1-\alpha)^2}{72}$
UP	$\frac{(1-\alpha)\theta_i - \alpha\theta_i^{\frac{1-\alpha}{\alpha}}}{1-2\alpha}$	$\frac{(1+\alpha)((1-\alpha)\Theta_{(1)} - \alpha\left(\Theta_{(1)}\right)^{\frac{1-\alpha}{\alpha}})}{1-2\alpha}$	$\frac{(1-\alpha)^2(9\alpha^2+8\alpha+1)}{18(1+2\alpha)}$
DP	$\frac{1+\alpha-(1-\alpha)\theta_i^2}{2(1-(1-\alpha)\theta_i)}$	$\frac{1+\alpha-(1-\alpha)\Theta_{(2)}^2}{2\left(1-(1-\alpha)\Theta_{(2)}\right)} + \frac{\alpha\left(1+\alpha-(1-\alpha)\Theta_{(1)}^2\right)}{2\left(1-(1-\alpha)\Theta_{(1)}\right)}$	$\frac{36\alpha^5 Ln(\alpha)^2 + 36\alpha^3(\alpha-1)(\alpha+1)^2 Ln(\alpha)}{72(1-\alpha)^4} + \frac{(1-\alpha)(119\alpha^5-70\alpha^4-37\alpha^3-11\alpha^2-2\alpha+1)}{72(1-\alpha)^4}$

The model with the lowest variance is the new auction model *DV*. Then the *DV* auction model is the most attractive for Market Operator from the point of view of variance.

IV.3. Value at risk in Case 1

In this case the VaR can be expressed as

$$\begin{aligned} VaR_\beta^S &= inf\left\{k\epsilon\left[0,\frac{2\alpha}{3}\right]: P\left(X^S - \frac{2\alpha}{3} < k\right) \geq \beta\right\} \\ &= inf\left\{k\epsilon\left[0,\frac{2\alpha}{3}\right]: F^{X^S}\left(k + \frac{2\alpha}{3}\right) \geq \beta\right\} \end{aligned} \quad \textbf{(14)}$$

Replacing $F^{X^S}(x)$, the corresponding distribution function of X^S, into the expression 14, the VaR and RVaR for *VI* and *U\D* can be obtained. They are presented in Table 3.

Table 3.
Expected revenue, distribution function, VaR and RVaR of VI and U\D in Case 1.

	X^S	$F^{X^S}(x)$	in:	VaR^S_β	$RVaR^S_\beta$
VI	$\alpha\ \Theta_{(1)}$	$\left(\frac{x}{\alpha}\right)^2$	$[0,\alpha]$	$\alpha\left(\sqrt{\beta}-\frac{2}{3}\right)$	$\left(\frac{3}{2}\sqrt{\beta}-1\right)100$
$U\backslash D$	$\frac{\alpha\left(1+\Theta_{(2)}\right)}{2}$	$\frac{8\alpha x-3\alpha^2-4x^2}{\alpha^2}$	$\left[\frac{\alpha}{2},\alpha\right]$	$\alpha\left(\frac{1}{3}-\frac{\sqrt{1-\beta}}{2}\right)$	$\left(\frac{1}{2}-\frac{3\sqrt{1-\beta}}{4}\right)100$

Table 4.
VaR and RVaR of UnD and V I at confidence level 0.95 for some values of α in Case 1.

α	0.2	0.4	0.6	0.8
$VaR^{U\backslash D}_{0.95}$	0.044	0.089	0.133	0.177
$RVaR^{U\backslash D}_{0.95}$	33.23%	33.23%	33.23%	33.23%
$VaR^{VI}_{0.95}$	0.062	0.123	0.185	0.246
$RVaR^{V}_{0.95}$	46.2%	46.2%	46.2%	46.2%

For example, if focused on the *VI* auction model then the Market Operator pays an extra amount less tan 46.2% of the expected payment, with probability greater than 0.95. The model with the lowest VaR is *U\D* auction model. Then the UnD auction model is the most attractive for Market Operator from the point of view of value at risk.

IV.4. Value at risk in Case 2

In this VaR can be written as

$$VaR_\beta^S = inf\left\{k\epsilon\left[0,\tfrac{2}{3}(2\alpha+1)\right]: P\left(X^S-\tfrac{2}{3}(2\alpha+1)<k\right)\geq\beta\right\}$$
$$= inf\left\{k\epsilon\left[0,\tfrac{2}{3}(2\alpha+1)\right]: F^{X^S}\left(k+\tfrac{2}{3}(2\alpha+1)\right)\geq\beta\right\} \quad \textbf{(15)}$$

The VaR and RVaR can be estimated for *VI* and *DV* because $F^{X^S}(x)$ can be obtained, the distribution function of X^S (given in Table 5).

However in *UP* and *DP* is necessary to do a numerical computation to calculate the sample VaR because the corresponding distribution functions cannot be obtained. The sample VaR corresponds to the quantile $1-\beta$ of the variable X^S -$\frac{2}{3}(2\alpha+1)$ that we simulate 100000 times. The steps for the simulation, for each value of α, are: obtain 100000 uniformly distributed random realizations in [0; 1] of $\{\Theta_1,\Theta_2\}$, substitute these values in the corresponding Bayesian Nash equilibrium getting a sample of the variable X^S -$\frac{2}{3}(2\alpha+1)$ and, finally, calculate the quantile $1-\beta$ of the obtained sample.

Table 5.
Expected revenue, distribution function, VaR and RVaR of the main four auction models in Case 2.

	X^S	$F^{X^S}(x)$	in:	VaR_β^S	$RVaR_\beta^S$
VI	$(1-\alpha)\,\Theta_{(1)}+2\alpha$	$\left(\frac{x-2\alpha}{1-\alpha}\right)^2$	$[2\alpha,1+\alpha]$	$(1-\alpha)\left(\sqrt{\beta}-\frac{2}{3}\right)$	$\frac{(1-\alpha)100\left(\frac{3}{2}\sqrt{\beta}-1\right)}{2\alpha+1}$
DV	$\frac{(1-\alpha)\Theta_{(2)}}{2}+\frac{1+3\alpha}{2}$	$\frac{8(1+\alpha)x-4x^2}{(1-\alpha)^2}-\frac{(\alpha+3)(3\alpha+1)}{(1-\alpha)^2}$	$\left[\frac{1+3\alpha}{2},1+\alpha\right]$	$(1-\alpha)\left(\frac{1}{3}-\frac{\sqrt{1-\beta}}{2}\right)$	$\frac{(1-\alpha)50\left(1-\frac{3\sqrt{1-\beta}}{2}\right)}{2\alpha+1}$
UP	$\frac{(1+\alpha)((1-\alpha)\Theta_{(1)}-\alpha\left(\Theta_{(1)}\right)^{\frac{1-\alpha}{\alpha}})}{1-2\alpha}$	numerical simulation	$[0,1+\alpha]$	numerical simulation	numerical simulation
DP	$\frac{1+\alpha-(1-\alpha)\Theta_{(2)}^2}{2\left(1-(1-\alpha)\Theta_{(2)}\right)}+\frac{\alpha\left(1+\alpha-(1-\alpha)\Theta_{(1)}^2\right)}{2\left(1-(1-\alpha)\Theta_{(1)}\right)}$	numerical simulation	$\left[\frac{(1+\alpha)^2}{2},1+\alpha\right]$	numerical simulation	numerical simulation

Thus, the results reported in Table 6 were obtained.

Table 6.
VaR and RVaR of the main four auction models at confidence level 0.95 for some values of α

α	0.2	0.4	0.6	0.8
$VaR^{DV}_{0.95}$	0.177	0.133	0.089	0.044
$RVaR^{DV}_{0.95}$	18.99%	11.08%	6.04%	2.56%
$VaR^{DP}_{0.95}$	0.211	0.168	0.116	0.059
$RVaR^{DP}_{0.95}$	22.61%	13.98%	7.92%	3.44%
$VaR^{VI}_{0.95}$	0.246	0.185	0.123	0.062
$RVaR^{V}_{0.95}$	26.4%	15.4%	8.4%	3.55%
$VaR^{UP}_{0.95}$	0.265	0.199	0.133	0.066
$RVaR^{UP}_{0.95}$	28.41%	16.61%	9.07%	3.84%

For example, in the case of the *DP* auction model with α = 0.4 the Market Operator pays an extra amount less than 13.98% of the expected payment, with probability greater than 0.95. The model with the lowest VaR is the *DV* auction model. Then the *DV* auction model is the most attractive for Market Operator from the point of view of value at risk.

Remark 5. The Table 6 contains values obtained by theoretical distributions and values obtained by simulation. The analytical data are the same, with the precision chosen, if all these values were approached by a simulation.

V. Conclusions

The present paper analyzes a parametric family of auction models called General Auction Models (*GAM*) defined in Alonso and Tejada (2010), which includes, as particular cases the Uniform, the Discriminatory and the Vickrey auction models. It assumes that bidders are symmetric, have identical production capacity, are risk neutral and have independent private costs.

Alonso and Tejada (2010) obtain that there is a unique Bayesian Nash equilibrium for every auction model belonging to *GAM*. All auctions models belonging to *GAM* have efficient equilibria and do not incur in ex post loss problems. These auction models are identical in terms of the revenue but differ in terms of loss risk.

The present paper proposes the value at risk (VaR), a widely used risk measure in several economic contexts, as a possible criterion to differentiate between the auction models. Thus, it analyzes VaR on the Market Operator payment to suppliers at equilibrium and it obtains that auction model situated in the vertexes of *GAM* is more interesting for the Market Operator with respect to this criterion. With VaR the present paper measures how much the Market Operator can pay over expected payment at a certain probability.

In these vertexes, it finds the classic auction models plus a new auction model called *DV*. According to VaR criterion, the Market Operator should choose the auction model with lowest VaR in each case. When the demand is less than 1 (Case 1), the lowest VaR model is Uniform/Discriminatory auction model. However, when the demand is more than 1 (Case 2), the lowest VaR model is the new auction model DV. It is also possible to calculate doing numerical computation the sample VaR and the sample RVaR for all auction model belonging to *GAM*, because all Bayesian Nash equilibrium are known. All auctions belonging to *GAM* not situated in the vertexes of *GAM* that were simulated have a greater VaR than the models listed as the lowest VaR model in both cases.

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